Some properties of topological greyscale watersheds

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ABSTRACT

In this paper, we investigate topological watersheds.[1] For that purpose we introduce a notion of "separation between two points" of an image. One of our main results is a necessary and sufficient condition for a map G to be a watershed of a map F, this condition is based on the notion of separation. A consequence of the theorem is that there exists a (greedy) polynomial time algorithm to decide whether a map G is a watershed of a map F or not. We also show that, given an arbitrary total order on the minima of a map, it is possible to define a notion of "degree of separation of a minimum" relative to this order. This leads to another necessary and sufficient condition for a map G to be a watershed of a map F. At last we derive, from our framework, a new definition for the dynamics of a minimum.

Keywords: discrete topology, graph, watershed, dynamics, separation

1. INTRODUCTION

The watershed transform[2–8] of greyscale images is very popular as an important step of image segmentation.[9–11] Nevertheless, most existing approaches have three drawbacks:
- No clear formal definition of watersheds is used. As a consequence, no properties of these watersheds may be established.
- The watershed algorithms produce a binary result, that is, they lose the greyscale information that is present in the original image. This information may be useful for further processing (*e.g.*, connection of corrupted contours).
- Most popular watersheds algorithms, based on the flooding paradigm, produce watersheds which are not necessarily on the most significant contours of the original image.[12]

We investigate a topological approach[1] which allows to precisely define a greyscale watershed transform as an ultimate "W-thinning", a W-thinning is a kind of thinning which preserves the "lower connected components" of the original image (see also[13]). Here, a greyscale image is considered as a map from the set of vertices of an arbitrary graph to the set of integers. This approach is very general (e.g. it applies to images of arbitrary dimensions) and it does keep track of the useful greyscale information. An algorithm was proposed for extracting such a watershed from a map. Nevertheless, at this time, no general properties of topological watersheds were proved.

In this paper we show that a topological watershed has several fundamental properties. We introduce a notion of "separation" between two points x and y: x and y are separated if the lowest altitude for joining x and y is strictly greater than the altitudes of both x and y. We establish four theorems which are based on this notion of separation:

1) Roughly speaking, we define a map G to be a separation of a map F if G is lower than F and if, whenever x and y are separated for F (by an altitude k), then x and y are separated for G (by an altitude k). Our first theorem asserts that it is sufficient to consider the separation between vertices belonging to the minima of F and G to know whether G is a separation of F or not.

2) Our second theorem is a necessary and sufficient condition for a map G to be a W-thinning of a map F: a map G is a W-thinning of a map F if and only if G is a separation of F and the minima of G are "extensions" of minima of F. A consequence of this theorem is that there exists a (greedy) polynomial time algorithm to decide whether a map G is a watershed of a map F or not. This is an unexpected result because, in the classical

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framework of homotopy (and simple points), such an algorithm cannot exist.[14]
3) Our third theorem gives another characterization of W-thinnings which are proved to be equivalent to the maps which "preserve the lower connected components".
4) We show that, given an arbitrary total order on the minima of a map, it is possible to define a notion of "degree of separation of a minimum" relative to this order. Loosely speaking, we establish that a map G is a separation of a map F if and only if G preserves the degree of separation of each minimum of F. Combined with our second theorem, this leads to another necessary and sufficient condition for a map G to be a watershed of a map F. At last, we propose a new definition of the dynamics.[15, 16]

An extended version of this paper will show the link between topological watersheds and minimum spanning trees.[17] Forthcoming related papers will include properties of the dynamics,[18] a new "emergence" paradigm for extracting topological watersheds,[19] and quasi-linear time algorithms for topological watersheds.[20, 21]

2. BASIC DEFINITIONS

We define a *finite graph* as a pair (E, Γ) where E is a finite set of elements called *vertices* or *points* and Γ is a map from E to $\mathcal{P}(E)$, $\mathcal{P}(E)$ being the family composed of all subsets of E. If $y \in \Gamma(x)$, we say that y is *adjacent to* x. If $X \subseteq E$ and $y \in \Gamma(x)$ for some $x \in X$, we say that y is adjacent to X.
The graph (E, Γ) is *symmetric* if, for all x and y in E, we have $x \in \Gamma(y)$ whenever $y \in \Gamma(x)$; (E, Γ) is *reflexive* if, for any x in E, we have $x \in \Gamma(x)$.
Let $X \subseteq E$, a *path in* X is a sequence $\pi = x_0, ..., x_k$ such that $x_i \in X$, $i = 0, ..., k$, and $x_i \in \Gamma(x_{i-1})$, $i = 1, ..., k$. We also say that π is a *path from* x_0 *to* x_k. We say that X is *connected* if, for any x and y in X, there exists a path from x to y in X. We say that $Y \subseteq E$ is a *connected component of* $X \subseteq E$, if $Y \subseteq X$, Y is connected, and Y is maximal for these two properties. If $X \subseteq E$, we write $\overline{X} = \{x \in E;\ x \notin X\}$.

In the sequel of this paper, (E, Γ) will denote a finite reflexive and symmetric graph. For simplicity, we will furthermore assume that E is connected. All notions and properties may be easily extended for non-connected graphs.

Let $X \subseteq E$ and let $x \in X$. We say that the point x is:
- *a border point (for* X*)* if x is adjacent to at least one point of $\overline{X}$.
- *an inner point (for* X*)* if x is not a border point for X.
- *W-simple (for* X*)* if x is adjacent to exactly one connected component of $\overline{X}$.
- *separating (for* X*)* if x is adjacent to more than one connected component of $\overline{X}$.

We denote by $\mathcal{F}(E)$ the family composed of all maps on E to $\mathbb{Z}$.
Let $F \in \mathcal{F}(E)$ and let $k \in \mathbb{Z}$. We set $F_k = \{x \in E;\ F(x) \geq k\}$, F_k is the *cross-section of* F *at level* k.
Let $x \in E$ and let $k = F(x)$. We say that the point x is:
- *a border point (for* F*)* if x is a border point for F_k;
- *an inner point (for* F*)* if x is an inner point for F_k;
- *W-destructible (for* F*)* if x is W-simple for F_k;
- *separating (for* F*)* if x is separating for F_k.

Let $F \in \mathcal{F}(E)$ and let $p \in E$. We denote by $[F \setminus p]$ the element of $\mathcal{F}(E)$ such that $[F \setminus p](p) = F(p) - 1$ and $[F \setminus p](x) = F(x)$ for all $x \in E \setminus \{p\}$.

Let F and G be two elements of $\mathcal{F}(E)$. We say that G is a *W-thinning* of F, if:
i) $G = F$; or if
ii) there exists a map H which is a W-thinning of F and there exists a W-destructible point p for H such that $G = [H \setminus p]$.
We say that G is a *watershed of* F if G is a W-thinning of F and if there is no W-destructible point for G.

3	3	3	5	5	5	10	10	10	10	15	20	20
3	3	3	5	5	5	10	10	10	10	15	20	20
3	3	3	5	5	30	30	30	10	15	15	20	20
3	3	3	5	30	20	20	20	30	15	15	20	20
40	40	40	40	40	20	20	20	40	40	40	40	40
10	10	10	10	40	20	20	20	40	10	10	10	10
5	5	5	5	10	40	20	40	10	10	5	5	5
1	1	3	5	10	15	20	15	10	5	1	0	0
1	1	3	5	10	15	20	15	10	5	1	0	0
1	1	3	5	10	15	20	15	10	5	1	0	0

(a)

3	3	3	3	3	3	3	3	3	3	3	3	3
3	3	3	3	3	3	3	3	3	3	3	3	3
3	3	3	3	3	30	30	30	3	3	3	3	3
3	3	3	3	30	1	20	0	30	3	3	3	3
30	30	30	30	1	1	20	0	0	30	30	30	30
1	1	1	1	1	1	20	0	0	0	0	0	0
1	1	1	1	1	1	20	0	0	0	0	0	0
1	1	1	1	1	1	20	0	0	0	0	0	0
1	1	1	1	1	1	20	0	0	0	0	0	0
1	1	1	1	1	1	20	0	0	0	0	0	0

(b)

Figure 1. (a) original image, (b) a watershed of (a)

In Fig. 1 (a), a map F on E is depicted, here E is a subset of $\mathbb{Z}^2$ (a rectangle). We consider the map Γ induced by the well-known "4-adjacency relation".[22] A watershed of F (relative to (E, Γ)) is shown Fig. 1 (b).

Let F and K be elements of $\mathcal{F}(E)$ such that $K \leq F$. We say that G is a *W-thinning of F constrained by K* if G is a W-thinning of F such that $K \leq G$. We say that G is a *watershed of F constrained by K* if G is a W-thinning of F constrained by K and if any W-thinning H of G, with $H \neq G$, is such that the property $K \leq H$ is not true.

If x and y are two points of E, we denote by $\Pi(x, y)$ the set composed of all paths from x to y in E.
Let $F \in \mathcal{F}(E)$. If π is a path in E, we set $F(\pi) = Max\{F(z);$ for all points z appearing in $\pi\}$.
If x and y are two vertices of (E, Γ), we set $F(x, y) = Min\{F(\pi); \pi \in \Pi(x, y)\}$, $F(x, y)$ is the *pass value for F between x and y*. If X and Y are two subsets of E, the *pass value for F between X and Y* is defined by $F(X, Y) = Min\{F(x, y);\ x \in X, y \in Y\}$.

Let $F \in \mathcal{F}(E)$ and let $k \in \mathbb{Z}$. A connected component C of $\overline{F_k}$ is said to be a *(regional) minimum (for F)* if $C \cap \overline{F_{k-1}} = \emptyset$. We denote by $\mathcal{M}(F)$ the set composed of all minima of F.

Let $F \in \mathcal{F}(E)$. We say that $X \subseteq E$ is a *flat zone for F* if X is connected, $F(x) = F(y)$ for all x, y in X, and X is maximal for these two properties. If X is a flat zone, we denote by $F(X)$ the *altitude of X*, *i.e.*, we have $F(X) = F(x)$ for any $x \in X$.

Observe that a minimum is necessarily a flat zone. Furthermore a flat zone X is a minimum for F if and only if any point $x \in \overline{X}$ which is adjacent to X is such that $F(x) > F(X)$.

3. SEPARATION

We introduce the notion of separation which plays a key role in our framework.

Let $F \in \mathcal{F}(E)$ and let x and y be two points in E.
We say that x and y are *separated (for F)* if $F(x, y) > Max\{F(x), F(y)\}$.
We say that x and y are *k-separated (for F)* if x and y are separated and $k = F(x, y)$.
We say that x and y are *linked (for F)* if x and y are not separated, in other words x and y are linked if $F(x, y) = Max\{F(x), F(y)\}$.

We say that x *dominates* y *(for F)* if $F(x,y) = F(x)$, in other words, x dominates y if there is a path π in $\Pi(x,y)$ such that $F(x) \geq F(z)$, for all z in π.
We denote by $\Lambda(F)$ the relation $\Lambda(F) = \{(x,y) \in E \times E;\ x \text{ dominates } y\}$. We also set $\Lambda(x,F) = \{y \in E;\ (x,y) \in \Lambda(F)\}$.

We observe that:
i) Two points x and y are linked for F if and only if $y \in \Lambda(x,F)$ or $x \in \Lambda(y,F)$.
ii) If $y \in \Lambda(x,F)$ and $x \in \Lambda(y,F)$, then we have $F(x) = F(y)$. The converse is, in general, not true.
iii) The relation $\Lambda(F)$ is a preorder, *i.e.*, $\Lambda(F)$ is a reflexive and transitive relation.
iv) For any $x \in E$, there is at least one minimum X for F such that $X \subseteq \Lambda(x,F)$.
v) A subset X of E is a minimum for F if and only if, for all points x in X, $\Lambda(x,F) = X$.
vi) A point x of E belongs to a minimum for F if and only if, for all $y \in \Lambda(x,F)$, $x \in \Lambda(y,F)$.
vii) If X and Y are two distinct minima for F, then, for all $x \in X$, $y \in Y$, x and y are separated.
viii) Let x and y be two points belonging to the same flat zone for F. Then, for each point z of E, we have $F(x,z) = F(y,z)$. The converse is, in general, not true.

Property 1: *Let x and y be two points which are k-separated for F. If x dominates z, then z and y are k-separated.*

Proof:
Let π_1 be a path from x to y such that $F(x,y) = k$, thus $k > Max\{F(x), F(y)\}$. Let π_2 be a path from x to z such that $F(x,z) = F(x)$. We denote by π_2^{-1} the sequence obtained by reversing π_2.
The path $\pi_3 = \pi_2^{-1} \cdot \pi_1$ is a path from z to y such that $F(\pi_3) = k$, thus, we must have $F(z,y) \leq k$.
Suppose π_4 is a path from z to y. The path $\pi_5 = \pi_2 \cdot \pi_4$ is a path from x to y such that $F(\pi_5) = Max[F(x), F(\pi_4)]$. We must have $F(\pi_4) \geq k$ otherwise we would have $F(x,y) < k$, thus, $F(z,y) \geq k$. Therefore we have $F(z,y) = k$ and, since $F(z) \leq F(x)$, $F(z,y) > Max\{F(z), F(y)\}$. $\square$

Property 2: *Let $F \in \mathcal{F}(E)$ and let $k \in \mathbb{Z}$. Two points x and y belong to the same component of $\overline{F_k}$ if and only if $F(x,y) < k$.*

Proof:
We have $\overline{F_k} = \{x \in E;\ F(x) < k\}$. Thus two points x and y belong to the same component of $\overline{F_k}$ if and only if there is a path π from x to y such that $F(\pi) < k$. $\square$

Property 3: *Let $F \in \mathcal{F}(E)$ and let $k \in \mathbb{Z}$. Two points x and y belong to distinct components of $\overline{F_k}$ if and only if $F(x,y) \geq k > Max[F(x), F(y)]$.*

Proof:
i) If x and y belong to distinct components of $\overline{F_k}$, then $F(x,y) \geq k$ (Prop. 2), furthermore, since x and y are in $\overline{F_k}$, we must have $F(x) < k$ and $F(y) < k$.
ii) If $F(x) < k$ and $F(y) < k$, then x and y belong to some components of $\overline{F_k}$. Furthermore, if $F(x,y) \geq k$, these components must be distinct (Prop. 2). $\square$

The following property is a direct consequence of properties 2 and 3:

Property 4: *Let $F \in \mathcal{F}(E)$. Two points x and y are k-separated for F, if and only if:*
i) x and y belong to the same component of $\overline{F_{k+1}}$; and
ii) x and y belong to distinct components of $\overline{F_k}$.

Property 5: *Let $F \in \mathcal{F}(E)$ and let p be a separating point, we set $k = F(p)$. Let X and Y be two distinct components of $\overline{F_k}$ adjacent to p. Then any x in X and any y in Y are k-separated.*

Proof: There exist two points $x' \in X$ and $y' \in Y$ which are adjacent to p, thus $\pi = x', p, y'$ is a path. Furthermore, there exists a path π_X in X from x to x' and a path π_Y in Y from y' to y. The path $\pi' = \pi_X \cdot \pi \cdot \pi_Y$ is a path from x to y such that $F(\pi') < k+1$. Therefore, x and y belong to the same component of $\overline{F_{k+1}}$. Since x and y belong to distinct components of $\overline{F_k}$, x and y are k-separated (Prop. 4). $\square$

Let F and G be two elements of $\mathcal{F}(E)$ such that $G \leq F$. We say that G *is a separation of* F if, for all x and y in E, if x and y are k-separated for F, then x and y are k-separated for G.

The following theorem asserts that it is sufficient to consider minima of F for testing whether G is a separation of F or not.

Theorem 6 (restriction to minima): *Let F and G be two elements of $\mathcal{F}(E)$ such that $G \leq F$. The map G is a separation of F if and only if, for all distinct minima X, Y in $\mathcal{M}(F)$, $F(X,Y) = G(X,Y)$.*

Proof:
i) Suppose G is a separation of F and let X, Y be distinct minima in $\mathcal{M}(F)$. Let $F(X,Y) = k$. For all $x \in X$, $y \in Y$, x and y are k-separated for F, hence they are k-separated for G.
Thus, $G(X,Y) = Min\{G(x,y);\ x \in X, y \in Y\} = k = F(X,Y)$.
ii) Suppose G is not a separation of F, *i.e.*, there exist two points x and y which are k-separated for F but not k-separated for G. If x and y are not k-separated for G, it means either $G(x,y) \neq k$, or $G(x,y) = Max\{G(x), G(y)\}$. Since $G \leq F$, in both cases, we must have $G(x,y) < k$. Let X and Y be two minima for F such that $X \subseteq \Lambda(x,F)$ and $Y \subseteq \Lambda(y,F)$, thus $F(\{x\},X) = F(x) < k$ and $F(\{y\},Y) = F(y) < k$.
By Prop. 1, we have $F(X,Y) = k$. Since $G(\{x\},X) \leq F(\{x\},X) < k$ and $G(\{y\},Y) \leq F(\{y\},Y) < k$, we have $G(X,Y) \leq Max\{G(X,\{x\}), G(x,y), G(\{y\},Y)\} < k$. Therefore, $G(X,Y) \neq F(X,Y)$. $\square$

4. MINIMA EXTENSIONS

If a map G is a separation of a map F, it may be seen that G may have more minima than F. Since our purpose is to study W-thinnings and since W-thinnings cannot generate new minima, we introduce the following notion.

Let F and G be two maps in $\mathcal{F}(E)$ such that $G \leq F$.
We say that G is a *minima extension* or a *m-extension of* F if there is a bijection $\epsilon : \mathcal{M}(F) \to \mathcal{M}(G)$ such that:
i) for all $X \in \mathcal{M}(F)$, $X \subseteq \epsilon(X)$; and
ii) for all $X \in \mathcal{M}(F)$, $F(X) = G[\epsilon(X)]$.

Property 7: *Let G be a m-extension of F and let ϵ be the corresponding bijection. Then, for each $X \in \mathcal{M}(G)$, $\epsilon^{-1}(X) = \{x \in X;\ F(x) = G(x)\}$.*

Proof: Let $X \in \mathcal{M}(G)$. We first observe that, if Y is a minimum for F, and if $Y \cap X \neq \emptyset$, then $Y = \epsilon^{-1}(X)$, otherwise $\epsilon(Y)$ and X would be distinct minima for G with non empty intersection.
We have $\epsilon^{-1}(X) \subseteq \{x \in X;\ F(x) = G(x)\}$. Let $x \in X$, with $F(x) = G(x)$. Suppose $x \notin \epsilon^{-1}(X)$. Let Y be a minimum for F such that $Y \in \Lambda(x,F)$. It is not possible that $Y = \epsilon^{-1}(X)$, otherwise there would be a path of constant altitude between x and Y and we would have $x \in \epsilon^{-1}(X)$. Thus, $Y \neq \epsilon^{-1}(X)$, and, by the preceding remark, we must have $Y \cap X = \emptyset$. Let $\pi = x_0, ..., x_l$ be a path from $x = x_0$ to $x_l \in Y$ and such that $F(\pi) = F(x)$. Let i the smallest value such that $x_i \notin X$. Since X is a minimum for G, we must have

$G(x_i) > G(X)$. Since $F(x_i) \geq G(x_i)$, we would have $F(x_i) > F(x)$ which contradicts $F(\pi) = F(x)$. $\square$

We observe that, if G is a m-extension of F, and H is a m-extension of G, then H is a m-extension of F. Furthermore, we have the two properties:

Property 8: *Let G and H be m-extensions of F. If $G \leq H$, then G is a m-extension of H.*

Proof: We denote by ϵ_G and ϵ_H the corresponding bijections relative to G and H, respectively. The bijection $\epsilon_G \circ \epsilon_H^{-1} : \mathcal{M}(H) \to \mathcal{M}(G)$ satisfies the condition ii) for m-extensions. Suppose the condition i) is not satisfied. It means there is a minimum X for H such that $X \not\subseteq \epsilon_G \circ \epsilon_H^{-1}(X)$, we set $Y = \epsilon_G \circ \epsilon_H^{-1}(X)$. Since $\epsilon_H^{-1}(X) \subseteq X$ and $\epsilon_H^{-1}(X) \subseteq Y$, we have $X \cap Y \neq \emptyset$. Since X is connected, it means there is a point $x \in X \cap \overline{Y}$ which is adjacent to Y. We have $H(x) = H(X) = G(Y)$. It is not possible that $G(x) < G(Y)$, otherwise Y would not be a minimum for G. But since $G \leq H$, we must have $G(x) = G(Y)$ and x would belong to Y. $\square$

Property 9: *Let G be a m-extension of both F and H. If $H \leq F$, then H is a m-extension of F.*

Proof: We denote by ϵ_F and ϵ_H the corresponding bijections relative to F and H, respectively. The bijection $\epsilon_H^{-1} \circ \epsilon_F : \mathcal{M}(F) \to \mathcal{M}(H)$ satisfies the condition ii) for m-extensions. Let $X \in \mathcal{M}(F)$. By Prop. 7, we have $\epsilon_H^{-1} \circ \epsilon_F(X) = \{x \in \epsilon_F(X);\ H(x) = G(x)\}$. We have $X \subseteq \epsilon_F(X)$. Furthermore, for all $x \in X$, $F(x) = G(x)$. Since $G \leq H \leq F$, we have $H(x) = G(x)$, for all $x \in X$. Thus $X \subseteq \epsilon_H^{-1} \circ \epsilon_F(X)$: the map $\epsilon_H^{-1} \circ \epsilon_F$ satisfies the condition i) for m-extensions. $\square$

Let F and G be two maps in $\mathcal{F}(E)$ such that $G \leq F$. We say that G is a *m-cover of* F if any minimum X for G contains at least one minimum Y for F such that $G(X) = F(Y)$.
We say that G is a *strong separation of* F if G is both a separation of F and a m-cover of F.

Property 10: *If G is a strong separation of F, then G is a m-extension of F.*

Proof: Suppose G is a strong separation of F.
i) Let X be a minimum for G. There exists a minimum Y for F such that $Y \subseteq X$ and $G(X) = F(Y)$. Suppose x and x' are two elements of X such that $x \in Y$, $x' \in Y'$, with $Y' \in \mathcal{M}(F)$. Then we must have $Y = Y'$, otherwise x and x' would be separated for F and linked for G. Thus, any minimum X for G contains a unique minimum Y for F, and furthermore $G(X) = F(Y)$.
ii) Let X be a minimum for F and let $x \in X$. Let Y be a minimum for G such that $Y \subseteq \Lambda(x, G)$. ¿From i), there is a unique minimum X' for F such that $X' \subseteq Y$. We must have $X = X'$, otherwise x and any element $x' \in X'$ would be separated for F but not separated for G. Thus any minimum for F is contained in a minimum for G. Of course, this minimum is unique. $\square$

5. THE STRONG SEPARATION THEOREM

We are now in position to prove the equivalence between W-thinnings and strong separations. Beforehand, we have to establish the following property.

Property 11:
i) Let G and H be strong separations of F. If $G \leq H$, then G is a strong separation of H.
ii) Let G be a strong separation of both F and H. If $H \leq F$, then H is a strong separation of F.

Proof:
i) The maps G and H are m-extensions of F (Prop. 10). By Prop. 8, G is also a m-extension of H. Thus, it is

sufficient to prove that G is also a separation of H.
Let X' and Y' be two distinct minima for H, let X, Y be the corresponding minima for F, and let X'', Y'' be the corresponding minima for G. Thus, $X \subseteq X' \subseteq X''$ and $Y \subseteq Y' \subseteq Y''$. We have $F(X,Y) = H(X,Y) = H(X',Y')$ (H is a separation of F) and $F(X,Y) = G(X,Y) = G(X',Y')$ (G is a separation of F). Thus, $H(X',Y') = G(X',Y')$. Therefore, by Th. 6, G is a separation of H.
ii) The map G is a m-extension of F and H (Prop. 10). By Prop. 9, H is also a m-extension of F. Thus, it is sufficient to prove that H is also a separation of F.
Let X and Y be two distinct minima for F, let X', Y' be the corresponding minima for H, and let X'', Y'' be the corresponding minima for G. Thus, $X \subseteq X' \subseteq X''$ and $Y \subseteq Y' \subseteq Y''$. We have $F(X,Y) = G(X,Y) = G(X',Y')$ (G is a separation of F) and $H(X,Y) = H(X',Y') = G(X',Y')$ (G is a separation of H). Thus, $F(X,Y) = H(X,Y)$. Therefore, by Th. 6, H is a separation of F. $\square$

Theorem 12 (strong separation): *Let F and G be two elements of $\mathcal{F}(E)$. The map G is a W-thinning of F if and only if G is a strong separation of F.*

Proof:
1) Let p be a W-destructible point for F.
i) Let X be a minimum for $[F \setminus p]$.
Suppose $p \in X$. We note that it is not possible that $X = \{p\}$, otherwise p would be an inner point for F. Thus X contains at least two points. We can see that $X \setminus \{p\}$ is necessarily a minimum for F and that we have $F[X \setminus \{p\}] = [F \setminus p](X)$. Suppose $p \notin X$. Then X is a minimum for F and, trivially, $F(X) = [F \setminus p](X)$. Thus, in any cases, $[F \setminus p]$ is a m-cover of F. By induction, if G is a W-thinning of F, then G is a m-cover of F.
ii) Suppose x and y are k-separated for F but not k-separated for $[F \setminus p]$. We observe that we must have $[F \setminus p](x,y) = k-1$. Thus, there is a path $\pi = x_0, ..., x_l$, with $x_0 = x$, $x_l = y$, and such that $[F \setminus p](\pi) = (k-1)$. Any path from x to y contains an elementary path from x to y, so we may suppose that π is elementary, *i.e.*, that all points appearing in π are distinct. We note that:
- there should be some x_i such that $x_i = p$, otherwise x and y would not be k-separated for F;
- for the same reason, we must have $F(p) = k$;
- we must have $F(x_j) < k$, for all $0 \leq j \leq l$ and $j \neq i$, otherwise, since π is elementary, we would have $[F \setminus p](\pi) \geq k$;
- we must have $0 < i < l$, otherwise, we would have $F(x,y) = Max\{F(x), F(y)\}$, and x and y would not be separated for F;
- x_{i-1} and x_{i+1} should be k-separated for F, otherwise we would have $F(x,y) < k$.
Thus, by Prop. 4, x_{i-1} and x_{i+1} would belong to distinct components of $\overline{F_k}$, and p would not be W-destructible for F.
By induction, if G is a W-thinning of F, then G is a separation of F.
2) Suppose G is a strong separation of F. Let H be a watershed of F constrained by G. If $H = G$ we are done. Suppose $H \neq G$.
The map H is a strong separation of F (first part of the proof), thus G is a strong separation of H (Prop. 11). Now, let p be a point such that $H(p) > G(p)$ and which is not W-destructible for H. Thus, p must be either an inner point or a separating point for H.
- Suppose p is a separating point for H. We set $k = H(p)$.
There should exist two points x and y which are adjacent to p and which belong to distinct components of $\overline{H_k}$. By Prop. 5, x and y are k-separated for H. The presence of the path $\pi = x, p, y$ implies that we would have $G(x,y) \leq (k-1)$. The points x and y would not be k-separated for G, thus p cannot be a separating point.
- Suppose p is an inner point. We denote by $D(p,G)$ the set composed of all points x such that there exists a descending path for G from p to x, *i.e.*, a path $x_O, ..., x_j$ such that $x_O = p$, $x_j = x$, and $G(x_i) \leq G(x_{i-1})$, $i = 1, ..., j$. It may be seen that $D(p,G)$ contains necessarily a minimum for G. Let X such a minimum. Since G is a m-extension of H, there is a minimum X' for H such that $X' \subseteq X$ and $H(X') = G(X') = G(X)$. There exists a descending path for G from p to X'. Let $\pi = x_0, ..., x_j$ be such a path, we have $x_0 = p$, and $H(x_j) = G(x_j) = H(X')$. Since $H(p) > G(p)$ there exists a largest number i such that $H(x_i) \neq G(x_i)$, this number satisfies $i < j$. Thus $H(x_i) > G(x_i)$, $H(x_{i+1}) = G(x_{i+1})$, and $G(x_{i+1}) \leq G(x_i)$ (since π is descending).

Therefore, we must have $H(x_i) > H(x_{i+1})$. In other words, x_i should be a border point for H.
Since, from the preceding argument, x_i cannot be a separating point, it means that x_i would be a W-destructible point for H such that $H(x_i) > G(x_i)$ which contradicts the fact that H is a watershed of F constrained by G. $\square$

Prop. 11 and Th. 12 lead to the following "confluence" property which shows that W-thinnings are related to greedy structures[23]:

Theorem 13 (confluence): *Let F, G, H be maps in $\mathcal{F}(E)$ such that G is a W-thinning of F and $G \leq H \leq F$.*
The map H is a W-thinning of F if and only if G is a W-thinning of H.

Let us consider the following recognition problem $\mathcal{P}$: given two maps F and $G \leq F$ in $\mathcal{F}(E)$, decide whether G is a W-thinning of F or not. By definition, G is a W-thinning of F if G may be obtained from F by iteratively lowering (by one) W-destructible points. If we directly use this definition for solving $\mathcal{P}$, we get an exponential method. By Th. 13, $\mathcal{P}$ may be solved by the following greedy method which is polynomial:
Set $H = F$;
i) arbitrarily select a point p which is W-destructible for H and which satisfies $H(p) > G(p)$;
ii) do $H = [H \setminus p]$.
Repeat i) and ii) until stability; G is a W-thinning of F if $H = G$, otherwise G is not a W-thinning of F.

The above confluence property does not hold in the framework of homotopic thinnings (by deformation retract,[24] or by collapse,[25] or by simple points removal[22]). A counter-example is the so-called Bing's house.[14] A 3D-cube C may be thinned till one point P, but it may also be thinned till a Bing's house B. We may have $P \subseteq B \subseteq C$, but B cannot be thinned till P. This very example shows that, in the general case, the above greedy method does not work for the recognition problem in the framework of homotopic thinnings.
Of course, W-thinnings preserve less topological characteristics than homotopic thinnings. Nevertheless, the confluence property ensures that arbitrary W-thinnings cannot get "stuck" in some configurations.

6. EXTENSIONS

In this section, we will prove the equivalence between W-thinnings and maps which preserve all "lower connected components".

Let X be a subset of E. We denote by $\mathcal{C}(X)$ the set composed of all connected components of $\overline{X}$.
Let X and Y be subsets of E such that $Y \subseteq X$. The *component map relative to* (X, Y), is the map ϵ from $\mathcal{C}(X)$ to $\mathcal{C}(Y)$ such that, for any $C \in \mathcal{C}(X)$, $\epsilon(C)$ is the connected component of $\overline{Y}$ which contains C.

Let F, G in $\mathcal{F}(E)$ such that $G \leq F$. Let $k \in \mathbb{Z}$. The *k-component map relative to* (F, G), is the component map ϵ_k relative to (F_k, G_k).
We say that G is an *extension of* F if, for any $k \in \mathbb{Z}$, ϵ_k is a bijection.

Property 14: Let G be an extension of F. If $X \in \mathcal{C}(F_k)$ and $Y \in \mathcal{C}(F_l)$, then $Y \subseteq X$ if and only if $\epsilon_l(Y) \subseteq \epsilon_k(X)$.

Proof:
i) If $Y \subseteq X$, we will have $\epsilon_l(Y) \cap \epsilon_k(X) \neq \emptyset$, and, since G is a map, we must have $\epsilon_l(Y) \subseteq \epsilon_k(X)$.
ii) Suppose $Y \not\subseteq X$. Without loss of generality, suppose $l \leq k$. Let Z be the element of $\mathcal{C}(F_k)$ which contains Y, we have $Z \neq X$. Since, from i), $\epsilon_l(Y) \subseteq \epsilon_k(Z)$, and since $X \neq Z$, $\epsilon_k(X)$ and $\epsilon_k(Z)$ are distinct: we must have $\epsilon_l(Y) \not\subseteq \epsilon_k(X)$. $\square$

Theorem 15: *Let F, G in $\mathcal{F}(E)$ such that $G \leq F$.*
The map G is a W-thinning of F if and only if G is an extension of F.

Proof:
i) Let p be a W-destructible point for F and let $F' = [F \setminus p]$. For any $k \neq F(p)$, $\mathcal{C}(F_k) = \mathcal{C}(G_k)$, trivially ϵ_k is a bijection. If $k = F(p)$, by the very definition of a W-destructible point, each component of $\mathcal{C}(F'_k)$ contains one component of $\mathcal{C}(F_k)$ and two distinct components of $\mathcal{C}(F_k)$ are contained in distinct components of $\mathcal{C}(F'_k)$. Thus, ϵ_k is a bijection. By induction, if G is a W-thinning of F, then G is an extension of F.
ii) Suppose G is an extension of F.
- Let X be a minimum for G and let $k = G(X) + 1$, we have $X \in \mathcal{C}_k(G)$. The set $Y = \epsilon_k^{-1}(X)$ is a component in $\mathcal{C}_k(F)$ and is a minimum for F otherwise there would exist $Z \in \mathcal{C}_{k-1}(F)$, with $Z \subsetneq Y$ and we would have $\epsilon_{k-1}(Z) \in \mathcal{C}_{k-1}(G)$, with $Z \subsetneq X$. We have also $Y \subseteq X$ and $F(Y) = G(X)$. Thus, G is a m-cover of F.
- Let X and Y be two minima for F and let $F(X, Y) = k$. Thus, X and Y are subsets of a component $Z \in \mathcal{C}_{k+1}(F)$. The set $\epsilon_{k+1}(Z)$ is in $\mathcal{C}_{k+1}(G)$, it contains X and Y (from Prop. 14), thus $G(X, Y) \leq k$. In a reverse way, if $Z \in \mathcal{C}(G_l)$ contains X and Y, $\epsilon_l^{-1}(Z) \in \mathcal{C}_l(F)$ contains X and Y (from Prop. 14), thus we must have $G(X, Y) = k$: G is a separation of F. From Th. 12, G is a W-thinning of F $\Box$

7. ORDERED MINIMA AND THE DYNAMICS

Let us consider the following recognition problem $\mathcal{P}'$: given two maps F and $G \leq F$ in $\mathcal{F}(E)$, decide whether G is a separation of F or not. If we directly apply the definition of a separation, we have to compute all the values $F(x, y)$, $x \in E$ and $y \in E$, check from these values which pairs of points are separated, and, for these pairs, check if $F(x, y) = G(x, y)$. We can say that n^2 pass values relative to F, with $n = |E|$ are used to solve $\mathcal{P}'$. Theorem 6 asserts that, in fact, it is sufficient to consider pass values between minima of F. Thus m^2 values relative to F are sufficient, with $m = |\mathcal{M}(F)|$. This shows that the above n^2 values contain some "redundant information". In this section, we will see that, again, these m^2 values contain redundant information and that only $m - 1$ values are necessary to solve $\mathcal{P}'$.

Let $F \in \mathcal{F}(E)$ and let $\mathcal{E}$ be a family composed of non empty subsets of E.
Let $\prec$ be an *ordering on* $\mathcal{E}$, *i.e.*, $\prec$ is a relation on $\mathcal{E}$ which is transitive and trichotomous (for any X, Y in $\mathcal{E}$, one and only one of $X \prec Y$, $Y \prec X$, $X = Y$ is true).
We denote by $X^\prec$ the element of $\mathcal{E}$ such that, for all $Y \in \mathcal{E} \setminus \{X^\prec\}$, $X^\prec \prec Y$.
Let $X \in \mathcal{E}$. The *pass value of X for* $(F, \prec)$ is the number $F(X, \prec)$ such that:
- If $X = X^\prec$, then $F(X, \prec) = \infty$; and
- If $X \neq X^\prec$, then $F(X, \prec) = Min\{F(X, Y);$ for all $Y \in \mathcal{E}$ such that $Y \prec X\}$.

Theorem 16 (ordered minima): *Let F, G be elements of $\mathcal{F}(E)$ such that $G \leq F$.*
Let $\prec$ be an ordering on the minima of F. The map G is a separation of F if and only if, for each minimum X for F, we have $F(X, \prec) = G(X, \prec)$.

Proof: By Th. 6, if G is a separation of F, then $F(X, \prec) = G(X, \prec)$.
Suppose $G \leq F$ is not a separation of F. By Th. 6, it means that there exist two distinct minima for F, say X and Y, such that $F(X, Y) \neq G(X, Y)$. We set $k = F(X, Y)$, thus $G(X, Y) < k$.
By Prop. 4, there exist two distinct components X' and Y' of $\overline{F_k}$ such that $X \subseteq X'$, $Y \subseteq Y'$. Furthermore, there exists a component C of $\overline{F_{k+1}}$ such that $X' \subseteq C$ and $Y' \subseteq C$.
Let X'' (resp. Y'') be the minimum for F which is a subset of X' (resp. Y') such that $X'' \prec Z$ (resp. $Y'' \prec Z$), for all $Z \in \mathcal{M}(F)$, $Z \subseteq X'$ and $Z \neq X''$ (resp. $Z \subseteq Y'$ and $Z \neq Y''$). By Prop. 2 and 4, we have $F(X, X'') < k$, $F(Y, Y'') < k$, and $F(X'', Y'') = k$.
Since $G(X'', Y'') \leq Max\{G(X'', X), G(X, Y), G(Y, Y'')\}$, and since $G \leq F$, we have $G(X'', Y'') < k$.
Without loss of generality, suppose $Y'' \prec X''$. We observe that, since all minima Z for F such that $F(X'', Z) < k$ satisfy $X'' \prec Z$, we must have $F(X'', \prec) \geq k$. Furthermore, since $Y'' \prec X''$, we have $F(X'', \prec) \leq k$. The result

is $F(X'', \prec) = k$.
But $G(X'', \prec) < k$, which follows from $G(X'', Y'') < k$ and $Y'' \prec X''$. ¿From this we conclude that $F(X'', \prec) \neq G(X'', \prec)$. $\square$

The above definition of the pass value of a minimum leads to a new notion of dynamics the definition of which is given below. In a forthcoming paper,[18] it will be shown that this notion "encodes more topological features" than the original one.[15]

Let $F \in \mathcal{F}$. Let $\prec$ be an ordering on $\mathcal{M}(F)$. We say that $\prec$ is an *altitude ordering on* $\mathcal{M}(F)$ if $X \prec Y$ whenever $F(X) < F(Y)$.
Let $\prec$ be an altitude ordering of $\mathcal{M}(F)$ and let X be a minimum for F. The *dynamics of* X *for* $(F, \prec)$ is the value $Dyn(X; F, \prec) = F(X, \prec) - F(X)$.

In Fig. 2, a topological watershed (b) of the original image (a) is represented. The minima of the watershed (c) illustrate the well-known over-segmentation problem. Using the methodology introduced in mathematical morphology[15] and our notions, we can extract all the minima which have a dynamics (according to an altitude ordering) greater than a given threshold (here 20), and suppress all others with a geodesic reconstruction . We obtain the image (d), the watershed (e), and the minima (f).

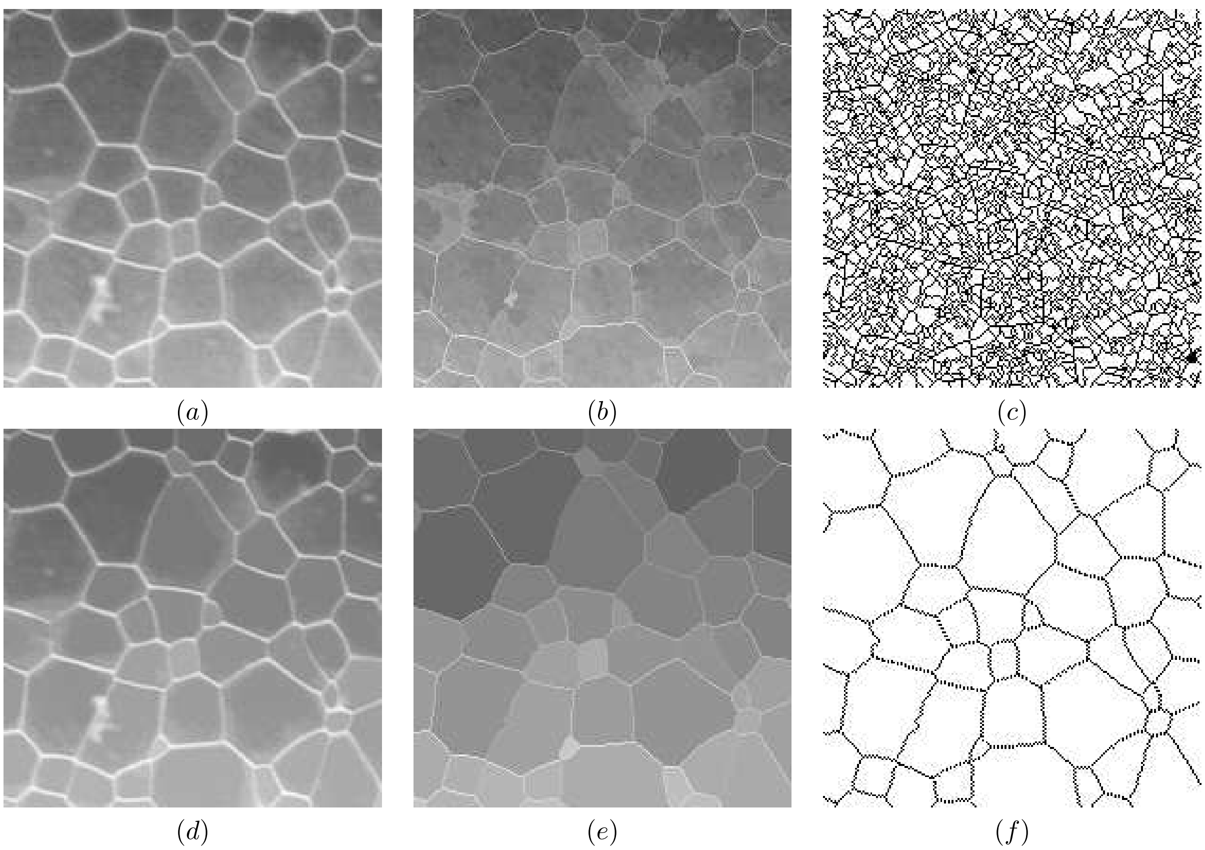

Figure 2. (a): original image, (b): a topological watershed of (a), (c): the minima of (b), (d): a filtering of (a) with ordered dynamics, (e): a topological watershed of (d), (f): the minima of (e).

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